

A new quantum $so(2,2)$ algebra

Francisco J. Herranz¹

Departamento de Física
Escuela Politécnica Superior
Universidad de Burgos
E-09006 Burgos, Spain

Abstract

By starting from the non-standard quantum deformation of the $sl(2, \mathbf{R})$ algebra, a new quantum deformation for the real Lie algebra $so(2, 2)$ is constructed by imposing the former to be a Hopf subalgebra of the latter. The quantum $so(2, 2)$ algebra so obtained is realized as a quantum conformal algebra of the $(1 + 1)$ Minkowskian spacetime. This Hopf algebra is shown to be the symmetry algebra of a time discretization of the $(1 + 1)$ wave equation and its contraction gives rise to a new $(2 + 1)$ quantum Poincaré algebra.

1 Introduction

The non-standard quantum deformation of $sl(2, \mathbf{R}) \simeq so(2, 1)$ [1], here denoted $U_z(sl(2, \mathbf{R}))$, has been the starting point in the obtention of non-standard quantum algebras in higher dimensions. In particular, by taking two copies of $U_z(sl(2, \mathbf{R}))$ and applying the same procedure as in the standard (Drinfel'd–Jimbo) case [2], a quantum $so(2, 2)$ algebra has been obtained in [3], while the corresponding deformation for $so(3, 2)$ has been found in [4]. These quantum algebras have been realized as deformations of conformal algebras for the Minkowskian spacetime. Furthermore, by following either a contraction approach [3] or a deformation embedding method [5], non-standard quantum deformations for other Lie algebras have been deduced; amongst them it is remarkable the appearance of a non-standard quantum Poincaré

¹ Contribution to the Proceedings of the ‘Quantum Theory and Symmetries’ (Goslar, 18-22 July 1999) (World Scientific, 2000), edited by H.-D. Doebner, V.K. Dobrev, J.-D. Hennig and W. Luecke.

algebra, which can be considered as a conformal quantum algebra for the Carroll spacetime, or alternatively as a null-plane quantum Poincaré algebra [5]. All these results are summarized in the following diagram where the vertical arrows indicate the corresponding contractions leading to Poincaré algebras:

$$\begin{array}{ccccc}
U_z(sl(2, \mathbf{R})) & \longrightarrow & U_z(sl(2, \mathbf{R})) \oplus U_{-z}(sl(2, \mathbf{R})) \simeq U_z(so(2, 2)) & \longrightarrow & U_z(so(3, 2)) \\
\downarrow \varepsilon \rightarrow 0 & & & & \downarrow \varepsilon \rightarrow 0 \\
U_z(iso(1, 1)) & \longrightarrow & \text{Null-plane Poincaré algebra} & \longrightarrow & U_z(iso(3, 1))
\end{array}$$

The aim of this contribution is to provide, starting again from $U_z(sl(2, \mathbf{R}))$, a new way in the obtention of non-standard quantum algebras. The first step is to construct a new non-standard quantum $so(2, 2)$ algebra which could be the cornerstone of further constructions in higher dimensions. The essential idea is to require that $U_z(sl(2, \mathbf{R}))$ remains as a Hopf subalgebra so that this approach can be seen as a kind of *complete* deformation embedding method. Next, a contraction limit gives rise to a new $(2 + 1)$ quantum Poincaré algebra which contains a $(1 + 1)$ quantum Poincaré Hopf subalgebra:

$$U_z(sl(2, \mathbf{R})) \subset U_z(so(2, 2)) \xrightarrow{\varepsilon \rightarrow 0} U_z(iso(1, 1)) \subset U_z(iso(2, 1))$$

It is interesting to stress that such new quantum $so(2, 2)$ algebra is the symmetry algebra of a time discretization of the wave equation. Thus we recall in the next section the basic facts of the Lie algebra $so(2, 2)$ in a conformal basis as well as its relationship with the $(1 + 1)$ wave equation. The Hopf algebra structure deforming $so(2, 2)$, its role as a discrete symmetry algebra and its contraction to Poincaré are presented in the section 3.

2 Lie algebra $so(2, 2)$

Let us consider the real Lie algebra $so(2, 2)$ generated by H (time translations), P (space translations), K (boosts), D (dilations) and C_1, C_2 (special conformal transformations). In this basis $so(2, 2)$ is the Lie algebra of the group of conformal transformations of the $(1 + 1)$ Minkowskian spacetime. The Lie brackets of $so(2, 2)$ read

$$\begin{array}{lll}
[K, H] = P & [K, P] = H & [H, P] = 0 \\
[D, H] = H & [D, C_1] = -C_1 & [H, C_1] = -2D \\
[D, P] = P & [D, C_2] = -C_2 & [P, C_2] = 2D \\
[K, C_1] = C_2 & [K, C_2] = C_1 & [C_1, C_2] = 0 \\
[H, C_2] = 2K & [P, C_1] = -2K & [K, D] = 0.
\end{array} \tag{1}$$

Three subalgebras of $so(2, 2)$ are relevant for our purposes:

- $\{H, P, K\}$ which span the $(1 + 1)$ Poincaré algebra (first row in (1)).
- $\{D, H, C_1\}$ which give rise to $so(2, 1) \simeq sl(2, \mathbf{R})$ (second row in (1)).
- $\{D, P, C_2\}$ which also generate $so(2, 1) \simeq sl(2, \mathbf{R})$ (third row in (1)).

A vector field representation of $so(2, 2)$ in terms of the space and time coordinates (x, t) is given by

$$\begin{aligned} H &= \partial_t & P &= \partial_x & K &= -t\partial_x - x\partial_t & D &= -x\partial_x - t\partial_t \\ C_1 &= (x^2 + t^2)\partial_t + 2xt\partial_x & C_2 &= -(x^2 + t^2)\partial_x - 2xt\partial_t. \end{aligned} \quad (2)$$

The Casimir of the above Poincaré subalgebra is $E = P^2 - H^2$. The action of E on a function $\Phi(x, t)$ through the representation (2) (choosing for E the value zero) leads to the $(1 + 1)$ wave equation:

$$E\Phi(x, t) = 0 \quad \implies \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \Phi(x, t) = 0. \quad (3)$$

We shall say that an operator \mathcal{O} is a symmetry of the equation $E\Phi(x, t) = 0$ if \mathcal{O} transforms solutions into solutions, that is, $E\mathcal{O} = \Lambda E$ where Λ is another operator. The Lie algebra $so(2, 2)$ is the symmetry algebra of the wave equation: E commutes with $\{H, P, K\}$ and in the realization (2) the remaining generators verify

$$[E, D] = -2E \quad [E, C_1] = 4tE \quad [E, C_2] = -4xE. \quad (4)$$

3 Non-standard quantum $so(2, 2)$ algebra

We choose the $sl(2, \mathbf{R})$ subalgebra of $so(2, 2)$ spanned by $\{D, H, C_1\}$. Then we write in terms of these generators the non-standard quantum deformation of $sl(2, \mathbf{R})$ in the form introduced in [6] and denote τ the deformation parameter. This means that the classical r -matrix we are considering for $so(2, 2)$ is $r = -\tau D \wedge H$ (which is a solution of the classical Yang–Baxter equation). Now we look for a quantum $so(2, 2)$ algebra that keeps the quantum $sl(2, \mathbf{R})$ algebra as a Hopf subalgebra: $U_\tau(sl(2, \mathbf{R})) \subset U_\tau(so(2, 2))$. The resulting coproduct and commutation rules for $U_\tau(so(2, 2))$ are given by:

$$\begin{aligned} \Delta(H) &= 1 \otimes H + H \otimes 1 & \Delta(P) &= 1 \otimes P + P \otimes e^{\tau H} \\ \Delta(D) &= 1 \otimes D + D \otimes e^{-\tau H} & \Delta(C_1) &= 1 \otimes C_1 + C_1 \otimes e^{-\tau H} \\ \Delta(K) &= 1 \otimes K + K \otimes 1 - \tau D \otimes e^{-\tau H} P \\ \Delta(C_2) &= 1 \otimes C_2 + C_2 \otimes e^{-\tau H} + 2\tau D \otimes e^{-\tau H} K - \tau^2 D(D + 1) \otimes e^{-2\tau H} P \end{aligned} \quad (5)$$

$$\begin{aligned} [K, H] &= e^{-\tau H} P & [K, P] &= (e^{\tau H} - 1)/\tau & [H, P] &= 0 \\ [D, H] &= (1 - e^{-\tau H})/\tau & [D, C_1] &= -C_1 + \tau D^2 & [H, C_1] &= -2D \\ [D, P] &= P & [D, C_2] &= -C_2 & [P, C_2] &= 2D \\ [K, C_1] &= C_2 & [K, C_2] &= C_1 - \tau D^2 & [C_1, C_2] &= -\tau(DC_2 + C_2D) \\ [H, C_2] &= e^{-\tau H} K + K e^{-\tau H} & [P, C_1] &= -2K - \tau(DP + PD) & [K, D] &= 0. \end{aligned} \quad (6)$$

It can be checked that the universal quantum R -matrix for $U_\tau(sl(2, \mathbf{R}))$ [6] also holds for $U_\tau(so(2, 2))$. In our basis this element reads

$$\mathcal{R} = \exp \{ \tau H \otimes D \} \exp \{ -\tau D \otimes H \}. \quad (7)$$

The relationship between $U_\tau(so(2, 2))$ and a discretization of the wave equation can be established by means of the following differential-difference realization which under the limit $\tau \rightarrow 0$ gives the classical realization (2):

$$\begin{aligned}
H &= \partial_t & P &= \partial_x \\
K &= -x \left(\frac{e^{\tau \partial_t} - 1}{\tau} \right) - t e^{-\tau \partial_t} \partial_x & D &= -x \partial_x - t \left(\frac{1 - e^{-\tau \partial_t}}{\tau} \right) \\
C_1 &= (x^2 + t^2 e^{-\tau \partial_t}) \left(\frac{e^{\tau \partial_t} - 1}{\tau} \right) + 2xt \partial_x + \tau x \partial_x + \tau x^2 \partial_x^2 \\
C_2 &= -(x^2 + t^2 e^{-2\tau \partial_t}) \partial_x - 2xt \left(\frac{1 - e^{-\tau \partial_t}}{\tau} \right) + \tau t e^{-2\tau \partial_t} \partial_x.
\end{aligned} \tag{8}$$

The generators $\{H, P, K\}$ close a deformed Poincaré subalgebra (although not a Hopf subalgebra) whose Casimir is now $E_\tau = P^2 - \left(\frac{e^{\tau H} - 1}{\tau} \right)^2$. If we introduce the realization (8) then we find a time discretization of the wave equation on a uniform lattice with x as a continuous variable:

$$E_\tau \Phi(x, t) = 0 \quad \implies \quad \left\{ \frac{\partial^2}{\partial x^2} - \left(\frac{e^{\tau \partial_t} - 1}{\tau} \right)^2 \right\} \Phi(x, t) = 0. \tag{9}$$

Therefore the deformation parameter τ appearing within the discrete derivative in (9) can be identified with the time lattice constant. Furthermore the generators (8) are symmetry operators of (9) since they fulfill

$$\begin{aligned}
[E_\tau, X] &= 0 \quad \text{for } X \in \{H, P, K\} & [E_\tau, D] &= -2E_\tau \\
[E_\tau, C_1] &= 4(t + \tau + \tau x \partial_x) E_\tau & [E_\tau, C_2] &= -4x E_\tau.
\end{aligned} \tag{10}$$

Hence we conclude that $U_\tau(so(2, 2))$ is the symmetry algebra of the discrete wave equation (9). In this respect we recall that the symmetries of a discretization of the wave equation in both coordinates (x, t) on a uniform lattice were computed in [7], showing that they are difference operators which preserve the Lie algebra $so(2, 2)$ as in the continuous case. Therefore some kind of connection between the results of [7] and our quantum $so(2, 2)$ algebra should exist as it was already established for discrete Shrödinger equations and quantum algebras [8].

To end with, we work out the contraction from $U_\tau(so(2, 2))$ to a new quantum Poincaré algebra: $U_\tau(so(2, 2)) \rightarrow U_\tau(iso(2, 1))$. We apply to the Hopf algebra $U_\tau(so(2, 2))$ the Inönü–Wigner contraction defined by the map

$$H \rightarrow \varepsilon H \quad P \rightarrow P \quad K \rightarrow \varepsilon K \quad C_1 \rightarrow \varepsilon C_1 \quad C_2 \rightarrow C_2 \quad D \rightarrow D \tag{11}$$

together with a transformation of the deformation parameter: $\tau \rightarrow \tau/\varepsilon$. The limit $\varepsilon \rightarrow 0$ leads to the coproduct and commutators of $U_\tau(iso(2, 1))$:

$$\begin{aligned}
\Delta(H) &= 1 \otimes H + H \otimes 1 & \Delta(P) &= 1 \otimes P + P \otimes e^{\tau H} \\
\Delta(D) &= 1 \otimes D + D \otimes e^{-\tau H} & \Delta(C_1) &= 1 \otimes C_1 + C_1 \otimes e^{-\tau H} \\
\Delta(K) &= 1 \otimes K + K \otimes 1 & \Delta(C_2) &= 1 \otimes C_2 + C_2 \otimes e^{-\tau H} + 2\tau D \otimes e^{-\tau H} K
\end{aligned} \tag{12}$$

$$\begin{array}{lll}
[K, H] = 0 & [K, P] = (e^{\tau H} - 1)/\tau & [H, P] = 0 \\
[D, H] = (1 - e^{-\tau H})/\tau & [D, C_1] = -C_1 & [H, C_1] = 0 \\
[D, P] = P & [D, C_2] = -C_2 & [P, C_2] = 2D \\
[K, C_1] = 0 & [K, C_2] = C_1 & [C_1, C_2] = 0 \\
[H, C_2] = 2e^{-\tau H}K & [P, C_1] = -2K & [K, D] = 0.
\end{array} \tag{13}$$

The universal quantum R -matrix for $U_\tau(iso(2, 1))$ is also (7) so that it is formally preserved under contraction. Note also that the generators $\{D, H, C_1\}$ give rise to a $(1 + 1)$ quantum Poincaré subalgebra such that: $U_\tau(iso(1, 1)) \subset U_\tau(iso(2, 1))$.

Finally we remark that if we would have chosen the $sl(2, \mathbf{R})$ subalgebra spanned by $\{D, P, C_2\}$ instead of the one generated by $\{D, H, C_1\}$, then we would have obtained a quantum $so(2, 2)$ algebra with P as primitive generator (instead of H). This second choice would lead to a space discretization of the wave equation. Both quantum $so(2, 2)$ algebras would be algebraically equivalent by the interchanges $H \leftrightarrow P$ and $C_1 \leftrightarrow C_2$, however their contraction would lead to inequivalent quantum Poincaré algebras. A complete analysis of all these possibilities will be presented elsewhere.

Acknowledgment

This work was partially supported by Junta de Castilla y León, Spain (Project CO2/399).

References

- [1] C. Ohn, *Lett. Math. Phys.* **25**, 85 (1992).
- [2] E. Celeghini, R. Giachetti, E. Sorace, and M. Tarlini, *J. Math. Phys.* **32**, 1159 (1991).
- [3] A. Ballesteros, F.J. Herranz, M.A. del Olmo, and M. Santander, *J. Phys. A: Math. Gen.* **28**, 941 (1995).
- [4] F.J. Herranz, *J. Phys. A: Math. Gen.* **30**, 6123 (1997).
- [5] A. Ballesteros, F.J. Herranz, M.A. del Olmo, and M. Santander, *Phys. Lett. B* **351**, 137 (1995).
- [6] A. Ballesteros and F.J. Herranz, *J. Phys. A: Math. Gen.* **29**, L311 (1996).
- [7] J. Negro and L.M. Nieto, *J. Phys. A: Math. Gen.* **29**, 1107 (1996).
- [8] L.M. Nieto, J. Negro, F.J. Herranz, and A. Ballesteros, in *CRM Procs. and Lecture Notes*, eds. D. Levi and O. Ragnisco **25**, 325 (CRM, Montreal, 2000).